Teaching Logic

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These notes present the lessons I developed over the years to help beginning mathematics students to understand the logic in a proof. Professional mathematicians use these tools instinctively, often without realizing that most people have to think carefully to see how logic works. It is a brief, informal introduction to logic as it is used in construction of arguments. It provides the logic tools needed to understand a proof by contradiction, proof of the contrapositive or *reductio ad absurdum*.

The essential elements of predicate logic as presented here are the following.

- Predicates
- OR, AND, NOT
- IF…THEN
- Converse
- Contrapositive
- Quantifiers
- Arguments and proofs

Remarks on Teaching and Learning Experience

Students have few problems with the idea of a predicate as a statement that can have a truth-value attached to it. They also quickly absorb the use of AND, OF and NOT. After a quick explanation, they don't make any errors.

It is a different situation with implications. When it comes to IF…THEN, students stumble. They don't see an immediate connection between logic and proof. Neither the truth table for $A \to B$ nor the equivalence of $A \to B$ and $(NOT\ A)\ OR\ B$ is obvious to most of humanity. Both are major events on the intellectual landscape. For instance, students typically find it hard to see that $A \to B$ can have a truth-value independent of the truth-values of A and B. People often believe that if $A \to B$ is true then A must be true. In fact, most people believe that $A \to B$ implies the converse $B \to A$.

I have found it most effective to have two sessions for logic in the course with a break between "contrapositive" and "quantifiers".

The Goal

The goal of these lessons is that a student can correctly write negations and contrapositive of an implication with quantifiers. Such as:

For every ε, there exists an N such that for every n, if $n > N$ AND $\varepsilon > 0$ then $|x_n - L| < \varepsilon$

If the system $AX = B$ has no solution then there is a $1\times m$ matrix Y such that $YA = 0$ and $YB \neq 0$.

Predicates

A predicate is a statement that has the property that it can be true or false. For example "It is raining." and " x > 9 " are predicates; the first is true or false depending on the weather and the second is true or false depending on the value of x. Examples of Not-a-Predicate: (i) x + 7 (ii) My cat Felix. These can't be considered true or false; the first is a number, the second is an animal.

Predicate Exercises

[1] Which of these is a statement that can be assigned a truth-value

a. $1 + i$ is a root of $z^7 + 27z + 11$.
b. My car is brown.
c. $1 + 2 + 3 + 4 + 5$.
d. Apples.
e. The Senate of Canada is a useful institution.
f. $\int x^2 dx$.
g. If x is a real number then $x^2 \geq 0$.

[2] Parse the following sentences into units that can have a truth-value (i.e. can be true or false).

a. If I get up early and I study hard then either I will pass the test or I'll see a beautiful sunrise.

b. If $\int_a^b e^{-x}\, dx > 10$ and $b > 3a$ then $a > 6$ or $a < -1$.

Truth Tables – OR, AND, NOT

If P and Q are predicates then we can form a new predicate " P OR Q" which is true when P or Q or both are true and is false when both of P and Q are false. This can be recorded in a truth table. The first two columns run through the various possible truth-values for P and Q; the last column records the truth-value for the new predicate P OR Q.

P	Q	P OR Q
T	T	T
T	F	T
F	T	T
F	F	F

The new predicate P AND Q is true only if both of the predicates P and Q are true. Negation is written NOT P which is a new predicate that is true precisely when P is false. The truth tables for the connectives AND and NOT are shown at the right.

P	Q	P AND Q	NOT P
T	T	T	F
T	F	F	F
F	T	F	T
F	F	F	T

Exercises on OR, AND, NOT

[1] Assign a truth-value to each of the following.

(a) $\int x^2dx = x^2 + C$ OR $(7^2 < 50)$ (b) $\int x^2dx \neq x^2 + C$ OR $(7^2 < 50)$

(c) $\int x^2dx \neq x^2 + C$ AND $(7^2 < 50)$ (d) $\int x^2dx = x^2 + C$ AND $(7^2 < 50)$

[2] Define two predicates: G = Larry went to the dance, H = Lucy played in the band. For each of the following predicates built up from G and H determine what events in the lives of Lucy and Larry make each one true. A truth table may help.

(a) G OR H, (b) NOT(G OR H), (c) NOT(G OR (NOT H))

(d) (NOT G) AND H (e) (NOT G) AND (NOT H)

Identify pairs of predicates among (a) - (e) which are true and false in exactly the same circumstances. There will be one left over; formulate a new predicate using G, H, AND and NOT that is equivalent to the leftover predicate.

[3] Try now to formulate the general rules for taking the negative of OR and AND. Make up a formula for NOT (P OR Q) using NOT and AND. Also give a formula for NOT (P AND Q) using only NOT and OR. *CHECK* your formulas using truth tables.

[4] Suppose that z is a complex number. What is the relationship between the following two sentences? (See Question [3].)

(a) [NOT ($z + \frac{1}{z}$ is real)] OR [($|z| = 1$) OR (z is real)]

(b) NOT [($z + \frac{1}{z}$ is real) AND ($|z| \neq 1$)]OR (z is real)

[5] Does this suggest anything to you about how to prove the following?

if $z + \frac{1}{z}$ is real then either $|z| = 1$ or z is real

Warm-up exercises

For each pair of predicates decide if they say the same thing about P, Q, and R.

(P OR Q) OR R	P OR (Q OR R)
(P AND Q) AND R	P AND (Q AND R)
(P AND Q) OR R	P AND (Q OR R)

Logical Equivalence

Predicates X and Y are <u>logically equivalent</u> if they have the same truth-values in all circumstances. This is the same as saying they have the same truth table. Logically equivalent predicates can replace each other in any argument.

In the Warm-up, (P OR Q) OR R is false only when all of P, Q and R are false. The same is true for P OR (Q OR R), so these compound predicates have the same truth table. They are logically equivalent and in any argument we can replace one with the other. We can always change the order of OR's. The same is true for AND's.

For (P AND Q) OR R and P AND (Q OR R), take P false, Q false and R true. Then the first predicate is true and the second predicate is false. Thus these two compound predicates are not logically equivalent and we cannot substitute one for the other in an argument.

The exercises above lead to the equivalences:

NOT(P OR Q) is equivalent to (NOT P) AND (NOT Q)
NOT(P AND Q) is equivalent to (NOT P) OR (NOT Q)

Definitions

We write the statement "if P then Q" as $P \to Q$. So "if x is a real number then $x^2 \geq 0$" as:

$$(x\ is\ a\ real\ number) \to x^2 \geq 0$$

Each implication $P \to Q$ has two related implications.

(i) The *converse* of $P \to Q$ is the implication $Q \to P$.

(ii) The *contrapositive* of $P \to Q$ is the implication $NOT\ Q \to NOT\ P$.

The Truth Table for $P \to Q$

A great deal of mathematical argument is written as "if this then that". When should we consider such a statement to be true? Consider the following three statements about a plane region $\mathcal{R}$.

(a) If $\mathcal{R}$ is a square then $\mathcal{R}$ is a rectangle.
(b) If $\mathcal{R}$ is not a rectangle then $\mathcal{R}$ is not a square.
(c) If $\mathcal{R}$ is a rectangle then $\mathcal{R}$ is a square.

(i) Do these statements say different things?
(ii) Are these statements logically equivalent?
(iii) Does the meaning of (a) change if $\mathcal{R}$ is a square or $\mathcal{R}$ is a rectangle or $\mathcal{R}$ is a circle?

Take some time to think about these questions. Write out your answers.

From this exercise, it is easy to see that the truth table for $P \to Q$ begins as follows:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	?
F	F	?

What happens when P is false? Here students often think that if $P \to Q$ is true than also P is true. Since $P \to Q$ is built from predicates P and Q, we have to assign it a truth-value even if P is false. The previous exercise helps. The following partial truth table has unknowns for some entries; the letters a, b, c refer to the statements above about rectangles and squares.

P	Q	(a) $P \to Q$	(b) $(NOT\ Q) \to\ (NOT\ P)$	(c) $Q \to P$
T	T	T	u	T
T	F	F	F	m
F	T	x	v	F
F	F	y	T	n

If you believe that (a) (above) is equivalent to (b) [the contrapositive] then their truth table columns are the same. Hence you must have $y = T, x = v = m, u = T, n = T$. This fills in more of the table:

P	Q	(a) $P \to Q$	(b) $(NOT\ Q) \to (NOT\ P)$	(c) $Q \to P$
T	T	T	T	T
T	F	F	F	x
F	T	x	x	F
F	F	T	T	T

If you believe that (a) is not equivalent to (c) [the converse] then their columns must be different. Hence, $x \neq F$ so $x = T$ and we get:

P	Q	(a) $P \to Q$	(b) $(NOT\ Q) \to (NOT\ P)$	(c) $Q \to P$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Thus we can deduce the truth table for $P \to Q$ if we have already accepted that an implication $P \to Q$ is equivalent to its contrapositive $(NOT\ Q) \to\ (NOT\ P)$ and is not equivalent to its converse $Q \to P$.

The truth table of $P \to Q$ is:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that this suggests that an implication $P \to Q$ is a relatively weak statement since only one circumstance makes it false.

Exercises

In each case, state (a) the converse and (b) the contrapositive.

[1] If wishes were horses then beggars would ride

[2] If $x > 5$ the $x^2 > 25$

[3] If $x > 0$ or $y > 0$ then $x^2 + y^2 > 0$

[4] If $z + \frac{1}{z}$ is real then either z is real or $|z| = 1$

[5] If $\int_a^b e^{-x} dx > 10$ and $b > 3a$ then $a > 6$ or $a < -1$

[6] You must have a ticket to win. (First parse this into if…then)

[7] Consider the following statements (that were relevant to the 1990's):

(i) If Canada stays in NAFTA then there will be more jobs in the future.
(ii) If there are more jobs in the future then Canada stays in NAFTA.
(iii) If Canada stays in NAFTA then there will not be more jobs in the future.
(iv) If there will not be more jobs in the future then Canada has not stayed in NAFTA.
(v) If Canada has not stayed in NAFTA then there will not be more jobs in the future.

Each of these statements is an implication. Determine which pairs of these statements are logically equivalent; feel free to use words like *converse* and *contrapositive*.

Be careful to use only logic; no economics, sociology or politics allowed.

Negation of an Implication

The truth table of $P \rightarrow Q$ and its negation are as follows.

P	Q	$P \rightarrow Q$	$NOT\ (P \rightarrow Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

This says that $P \rightarrow Q$ is false exactly when P is true and Q is false. The negation is true exactly when P is true and Q is false. We could write this as $P\ AND\ (NOT\ Q)$. Check with a truth table that this is a correct logical equivalent of $NOT\ (P \rightarrow Q)$. Thus the negation of an implication is not another implication. This can make a proof of $P \rightarrow Q$ by contradiction seem strange, but it makes such a proof easier in the sense that when you assume that $P \rightarrow Q$ is not true, you get both P and $NOT\ Q$ to work with.

Arguments (without quantifiers)

[1] If Maria did not meet James last night, then either Maria got the award or James was out of town. If Maria did not get the award then James did not meet Maria last night and the award was presented at the hotel. If the award was presented at the hotel then either Maria got the award or James was out of town. But James did meet Maria last night and James was not out of town.

Did Maria get the award?

[2] Here is a theorem from an algebra course. It is about two polynomials, f and g, of degree n and distinct real numbers x_i.

THEOREM: If $f(x_i) = g(x_i)$ for $i = 0,1 \dots, n$ then $f = g$.

Proof: Suppose that $f \neq g$. Define a new polynomial $h(x) = f(x) - g(x)$. Then $h(x)$ is a non-zero polynomial of degree at most n. Also if $h(x) = f(x) - g(x)$ and $f(x_i) = g(x_i)$ for $i = 0,1 \dots, n$. Then $h(x)$ has at least $n + 1$ roots. If $h(x)$ is a non-zero polynomial of degree at most n then $h(x)$ has at most n roots. Hence $f(x_i) = g(x_i)$ for $i = 0,1 \dots, n$ is false. Therefore, if $f(x_i) = g(x_i)$ for $i = 0,1 \dots, n$ then $f = g$.

Rewrite this proof in logic (predicate) notation using the following predicates:

A: $f(x_i) = g(x_i)$ for $i = 0,1 \dots, n$
B: $f = g$
C: $h(x) = f(x) - g(x)$
D: $h(x)$ is a non-zero polynomial of degree at most n
E: $h(x)$ has at least $n + 1$ roots

Justify the claim that it is a valid proof.

[3] Suppose that P, Q, and R are predicates. Using truth tables or otherwise, show that $P \to (Q \to R)$ and $(P \to Q) \to R$ are **not** logically equivalent.

Quantifiers

There are two types of quantifiers used in mathematical statements. Sometimes we say: "There is a solution for $f(x) = 0$." This asserts that something exists so is called an *existential quantifier*. We also sometimes say something like: "Every real number has a real cube root." This says that something is true for all the elements in some set. It is called a *universal quantifier*.

The following exercises explore what these quantifiers mean and how the order of quantifiers can change the meaning of a sentence.

Examples:

[1] Identify the quantifiers in these sentences. Rewrite using the standard phrases "for all…" and "there exists a…".

a. Every cloud has a silver lining.
b. There is a red car in the parking lot.
c. The system $AX = B$ has a solution
d. Any matrix can be swept to reduced row echelon form.
e. Every polynomial of odd degree has a real root.
f. For each choice of B the system $AX = B$ has a solution.
g. No complex number has a negative modulus.
h. The system $AX = B$ does not have a solution.

[3] Consider the following two sentences.

a. Every day I read part of some book.
b. There is a book I read part of every day.

Do these sentences say the same thing?
For each, how would you prove that it is true or that it is false?

[2] These examples have several quantifiers each. Determine the circumstances that make each a true statement. What would I have to exhibit to prove it is true?

a. Every sunny day there is a chickadee that visits all my bird feeders.
b. For every polynomial $f(X)$ there is a bound r such that if $x > r$ then $f(x) \neq 0$.
c. There are days when not every cloud has a silver lining.

Some Formalities

Each quantifier has an associated set called its *universe*. So the universe of "For every polynomial" is the set of polynomials. Confining x to a specific set is necessary since it is hard to prove a statement true for all x with no restrictions that x be something.

Often the universe of a quantifier is clear from the context. Sometimes though you need to specify the universe for the variable. For example, you may need to be clear when a variable is real or complex, positive or negative. It can clarify your writing to mention the universe of a variable.

For example, in:

If A is a matrix then the row rank and column rank of A are equal.

Here the phrase "A is a matrix", establishes the universe of A and stops us from making the equal rank claim about A if A happens to be, say, an indefinite integral.

There is a notational convention for quantifiers. We write:

$\forall\, x \in S$ to mean "For all x in the set S"
$\exists\, x \in S$ to mean "For some x in the set S"

In these expressions the set S is the universe of the variable x.

For example, start with:

Every sunny day there is a chickadee that visits all my bird feeders

Define sets:
D = sunny days
C = chickadees
BF = my bird feers.

The sentence at the heart of this is $V(d, c, f)$ = "On day d, chickadee c visits my feeder f".

The full sentence says:

$$\forall d \in D, \exists c \in C, \forall f \in BF\ V(d, c, f)$$

Exercise:

[1] What does the following say: $\forall d \in D, \exists c \in C, \exists f \in BF\ V(d, c, f)$

There is one more idea to talk about. Consider the following statement.

If for some a the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible then $d \neq 0$.

This sentence is claiming something about b, c and d but not about a. In a mathematical sentence, there may be some variables governed by quantifiers and some not. The sentence is then making a claim about the non-quantified variables.

Order of quantifiers

The meaning of a sentence can change if the order of its quantifiers changes. Consider these statements about a matrix A.

a. For every choice of column B, there is a column X such that $AX = B$.
b. There is a column X such that for all columns B, $AX = B$.

These make vary different claims. The first says that for this matrix A every system of equations with A as coefficient matrix has a solution. This would be true, for example, if A was a square invertible matrix. The second statement says that all these different systems have the same solution. This is never true.

The order of quantifiers of the same type does not matter. The order of quantifiers of different types does matter.

Negation of Quantifiers

Consider a general universal quantifier statement: For all x in S, $P(x)$ is true. To prove this we would need an argument that establishes that $P(x)$ is true for every element of S. So to prove it is false, all we need is one element of S that makes $P(x)$ false. In other words:
"There is an x in S such that $P(x)$ is not true" is equivalent to "it is not true that for all x in S, $P(x)$ is true."

Formally then:

$$NOT\ \forall x \in S\ P(x) \equiv \exists x \in S\ NOT\ P(x)$$

Similarly, "it is not true that for some x in S, $P(x)$ is true" exactly when it is true that for all x in S, $P(x)$ is false. So formally

$$NOT\ \exists x \in S\ P(x) \equiv \forall x \in S\ NOT\ P(x)$$

Informally, when you move a negation inside or outside a quantifier, that quantifier toggles to the other type.

Our Goal.

The goal was to be able write the negations and contrapositives with quantifiers. Such as:

A. For every ε, there exists an N such that for every n, if $n > N$ AND $\varepsilon > 0$ then $|x_n - L| < \varepsilon$.

B. If the system $AX = B$ has no solution then there is a $1\times m$ matrix Y such that $YA = 0$ and $YB \neq 0$.

Here we go. We use everything we have learned above to do this including but not limited to $NOT(P \rightarrow Q) \equiv P\ AND\ NOT\ Q$

Statement A:

For every ε, there exists an N such that for every n, if $n > N$ AND $\varepsilon > 0$ then $|x_n - L| < \varepsilon$.

Negation of A:

For some ε, for all N for some n, NOT [if $n > N$ AND $\varepsilon > 0$ then $|x_n - L| < \varepsilon$].

= For some ε, for all N, for some n, [$n > N$ AND $\varepsilon > 0$ AND $|x_n - L| \geq \varepsilon$].

Statement of B:

If the system $AX = B$ has no solution then there is a $1\times m$ matrix Y such that $YA = 0$ and $YB \neq 0$.

Contrapositive of B:

If for all $1\times m$ matrices Y either $YA \neq 0$ or $YB = 0$ then the system $AX = B$ has a solution.

Exercise:
Write the negatives and contrapositives where possible for the following.

a. If I get up early and I study hard then either I will pass the test or I'll see a beautiful sunrise.
b. If $x \geq 0$ then for some y, we have $x = y^2$.
c. If for every B the system $AX = B$ has a unique solution then A is square and A is invertible.
d. For every $\epsilon > 0$ there is a δ such that if $|x - a| < \delta$ then $|f(x) - L| < \epsilon$
   (Formal definition of $\lim_{x \to a} f(x) = L$)

FROM HERE ON THE QUESTIONS ARE RECREATIONAL:

For [1] to [3] each question is a set of statements that claims to prove something.

But what does it prove?

[1] (a) No ducks waltz.
(b) No officers ever decline to waltz.
(c) All my poultry are ducks.

[2] (a) A plum-pudding, that is not really solid, is mere porridge.
(b) Every plum-pudding, served at my table, has been boiled in a cloth.
(c) A plum-pudding that is mere porridge is indistinguishable from soup.
(d) No plum-pudding are really solid, except what are served at my table.

[3] (a) I call no day "unlucky", when Robinson is civil to me.
(b) Wednesdays are always cloudy.
(c) When people take umbrellas, the day never turns out fine.
(d) The only days when Robinson is uncivil to me are Wednesdays.
(e) Everybody takes his umbrella with him when it is raining.
(f) My "lucky" days always turn out fine.

These arguments are taken from Lewis Carroll, Mathematical Recreations of Lewis Carroll, Vol.1, pp 121. QA 95.D6 V.1. The logician Charles Dodgson wrote Alice in Wonderland and Through the Looking Glass under the pseudonym Lewis Carroll but his day job was Professor at Christ Church, Oxford. There are many more arguments in the cited reference - be prepared for Victorian racism, sexism etc.

[4] Consider the statement:

This sentence contains contains exactly two errors.

According to our definition, is this a predicate?

[5] Parse into quantifiers, implications etc:

> Amongst all the fish flying around the gymnasium, there is one for which, in every Computer Science Class, there is a Physics major that knows the weight of the fish.

[6] Consider:

> For every tire in the library, there is a car in the parking lot such that if the tire fits the car then the car is red.

Explain why this statement is true if either (i) there is a red car in the parking lot or (ii) there are no tires in the library. Do not hesitate to look at the negation of this statement.